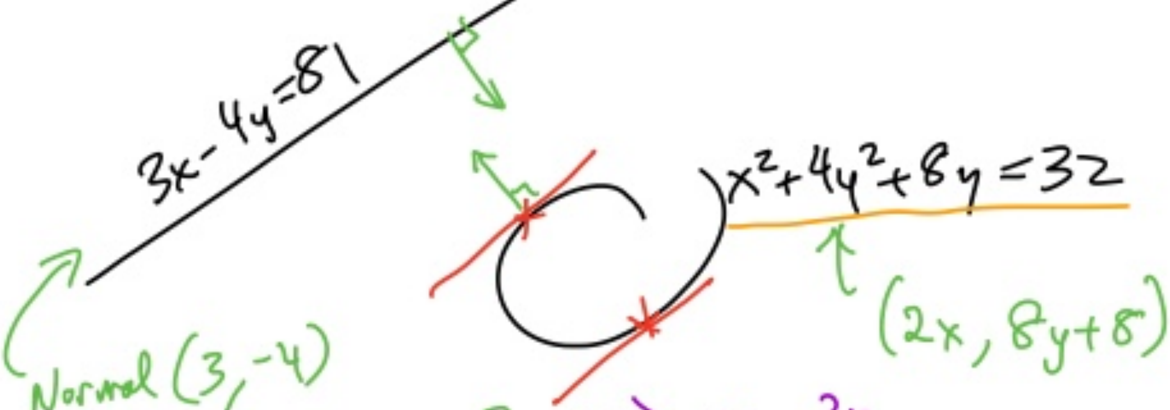
(2.29) Find all points on $x^2+4y^2+8y=32$ where the tangent line is $\parallel$ to $3x-4y=81$



$$\Rightarrow \begin{cases} 2x = c\cdot 3 \Rightarrow c = \frac{2x}{3} \\ 8y+8 = c\cdot(-4) \Rightarrow 8y+8 = \frac{2x}{3}(-4) \end{cases}$$

$$y+1 = \frac{-x}{3} \Rightarrow y = -1-\frac{x}{3}$$

$$\hookrightarrow x^2+4\left(-1-\frac{x}{3}\right)^2+8\left(-1-\frac{x}{3}\right)=32$$

$$x^2+4\left(1+\frac{x^2}{9}+\frac{2x}{3}\right)+-8-\frac{8x}{3}=32$$

$$9x^2+36+4x^2+\cancel{24x}-72-\cancel{24x}=32 \Rightarrow 13x^2=68 \Rightarrow x^2=\frac{68}{13}$$

$$\Rightarrow x = \pm\sqrt{\frac{68}{13}}$$

$$y = -1-\frac{x}{3} = \ldots$$

$$(x,y) = (\_,\_) \text{ or } (\_,\_)$$

Example Find the critical points of $f(x,y) = x^3-(x+2)y^2+y^3$. What types are they?

$$\nabla f = (0,0) \qquad \nabla f = \left(3x^2-y^2, -2(x+2)y+3y^2\right)$$

$$3x^2 = y^2 \Rightarrow y = \pm\sqrt{3}\,x$$

$$-2(x+2)y+3y^2=0$$

$$\hookrightarrow y\left(-2(x+2)+3y\right)=0$$

$$\Rightarrow y=0 \quad \boxed{\text{OR}} \quad -2(x+2)+3y=0$$

↳ If $y=0$
$x=0$
$(0,0)$ is a critical pt.

↳ $3y = 2(x+2)$

$$y = \frac{2}{3}(x+2)$$

$$\pm\sqrt{3}x = \frac{2}{3}x + \frac{4}{3}$$

$$\left(\pm\sqrt{3} - \frac{2}{3}\right)x = \frac{4}{3}$$

$$x = \frac{4}{3\left(\pm\sqrt{3} - \frac{2}{3}\right)}$$

$$\boxed{x = \frac{4}{\pm 3\sqrt{3} - 2}}$$

$$y = \pm\sqrt{3}\,x$$

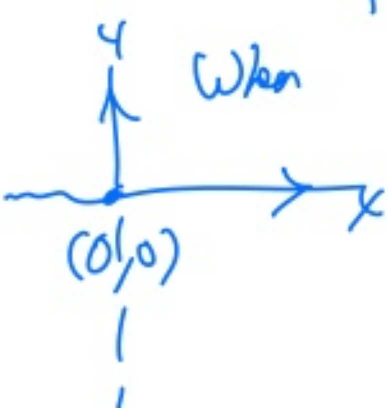
$$\boxed{y = \frac{\pm 4\sqrt{3}}{\pm 3\sqrt{3} - 2}}$$

Critical points: $(0,0)$, $\left(\frac{4}{3\sqrt{3}-2}, \frac{4\sqrt{3}}{3\sqrt{3}-2}\right)$,

$$\left(\frac{4}{-3\sqrt{3}-2}, \frac{+4\sqrt{3}}{+3\sqrt{3}+2}\right).$$

What types are these?

$$f(x,y) = x^3 - (x+2)y^2 + y^3.$$



When x-direction shape: when $y=0$

$$f(x,0) = x^3.$$

both up & down
→ saddle pt.

$$f(0,y) = -2y^2 + y^3$$

From desmos graph it looks like the other two critical points are a min and another saddle point.

Memories of Calc 1

Taylor series near $x=a$

$$f(x) = \underbrace{\underbrace{f(a) + f'(a)(x-a)}_{\text{tangent line}} + \frac{f''(a)}{2!}(x-a)^2}_{\text{"tangent parabola"}} + \cdots$$

If a is a critical pt of $f(x)$, then $f'(a)=0$

$$f(x) \approx f(a) + \frac{f''(a)}{2}(x-a)^2$$

2nd deriv. tells us the type of cr. point (if it's non zero).

- if $+$ cc up
- if $-$ cc down

Multivariable Calculus $f(x,y,\ldots)$

2nd derivative matrix
$$\begin{pmatrix} f_{xx} & f_{xy} & \cdots \\ f_{yx} & f_{yy} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

At a critical point $\nabla f = (0,0,\ldots)$

called the Hessian matrix.

symmetric $f_{xy} = f_{yx}$ (mixed partials commute)